

Projective relations for m-th root metric spaces

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Abstract

For Finsler spaces (M, F) endowed with m-th root metrics, we provide necessary and sufficient conditions in which they are projectively flat, or projectively related to Berwald/Riemann spaces. We also give a specific characterization for m-th root metrics spaces of Landsberg and of Berwald type.

Keywords: Finsler space, m-th root metric space, nonlinear connection, projective transformation, Berwald space, Landsberg space, Douglas space.

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1 Introduction

Finsler spaces with m-th root metric, defined by the fundamental function

$$F = \sqrt[m]{a_{i_1 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}} \quad (1)$$

have been introduced and studied in 1979 by Shimada, [14], and, in the case $m = 3$, by Matsumoto and Okubo, [10]. Most recently, they are taken into consideration by physicists as subject for a possible model of space-time, [12], [13], [15], [8], [7].

For quartic metrics ($m = 4$), a study of geodesics and of the related geometrical objects is made by S. Lebedev, [9], respectively, by V. Balan, S. Lebedev and the author, [5]. Also, Einstein equations for some relativistic models relying on such metrics are studied by V. Balan and the author in two papers, [3], [4]. Tensorial connections for such spaces have been recently studied by L. Tamassy, [16].

In the following, we shall give answers to the following questions:

- 1) In what conditions an m-th root metric space has rectilinear geodesics?
- 2) When do geodesics of such a space coincide with those of a Riemannian one?
- 3) In what conditions an m-root metric space is of Landsberg or of Berwald type?

The main idea is to use the Lagrange-type metric tensor

$$h_{ij} = a_{ij0 \dots 0}, \quad (2)$$

(which is nondegenerate), called in the following, the *polynomial flag metric* (V. Balan, [3]), whose coefficients h_{ij} are all polynomial functions in the directional variables y^i , instead of metric tensors traditionally used in the study of such spaces, namely, the usual Finsler metric, or Shimada's $a_{ij} = \frac{h_{ij}}{F^{m-2}}$ (which both contain m -th roots in their expressions). The expressions of corresponding geometrical objects and algorithms are thus simplified.

In Section 2, we express the equations of geodesics and the related geometrical objects (canonical nonlinear connection, Berwald linear connection etc.) in terms of the polynomial metric h_{ij} .

Section 3 is dedicated to m -th root metric spaces of Landsberg, respectively, of Berwald type; we show that the characterizing conditions written in terms of the polynomial flag metric h_{ij} look in similar way to their correspondents in terms of the usual Finsler metric, namely:

- that the Berwald connection should be h -metrical w.r.t. h_{ij} - for Landsberg spaces;
- that $h_{ijk|l} = 0$ (with respect to the canonical metrical connection of h_{ij}) - for Berwald spaces.

In the next section, we make a short review of (known) results regarding: projective relations between Finsler spaces, projective flatness, Finsler spaces projective to Riemannian ones, [6], [2], [17].

In Sections 5 and 6, we find the conditions upon the coefficients $a_{i_1 \dots i_m}(x)$ such that an m -th root metric space should be projectively flat, respectively, projective to a Riemannian space. By using the polynomial flag metric h , the determining algorithms reduce to simply identifying the coefficients of some polynomials.

In the last section, we analyze transformations

$$\bar{F}^{m'} = \alpha(x, y) F^m$$

and determine the conditions which make them become projective. As a consequence, we determine classes of Riemann-projective and projectively flat m -th root metrics.

As a remark, m -root metric functions F in (1) are not necessarily positive definite; still, under the assumption of nondegeneracy, we use the term of "Finsler" metric, [14].

2 m -th root metric spaces

Let M^n be a differentiable manifold of dimension n and class C^∞ , TM its tangent bundle and (x^i, y^i) the coordinates in a local chart on TM . Let F be the following function on M , [14]:

$$F = \sqrt[m]{a_{i_1 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}} \quad (3)$$

(with $a_{i_1 \dots i_m}$ symmetric in all its indices).

In the following, for a function $f = f(x, y)$, we shall denote by \cdot , \cdot and \cdot the partial derivatives w.r.t. x and y , respectively. Also, if N is a nonlinear connection on TM , we denote by \cdot its associate covariant derivative.

$$f_{;l} = \frac{\delta f}{\delta x^l} = \frac{\partial f}{\partial x^l} - N^r_l \frac{\partial f}{\partial y^r}, \quad f \in \mathcal{F}(TM).$$

Let T denote the m -th power of F :

$$T = F^m = a_{i_1 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}. \quad (4)$$

For the y -derivatives of F and T we shall omit the dot:

$$T_i = \frac{\partial T}{\partial y^i} = T_{\cdot i}, T_{ij} = T_{\cdot ij}, F_i = F_{\cdot i}, F_{ij} = F_{\cdot ij}$$

and we denote by null index transvection by y (e.g., $L_{i0} = L_{ij} y^j$).

Let

$$g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j} \quad (5)$$

the *usual Finsler metric* determined by F .

We denote by

$$h_{ij} = \frac{1}{m(m-1)} T_{ij} = a_{ij0 \dots 0}, \quad (6)$$

the *polynomial (flag) metric* attached to F .

Then we have:

$$h_{i0} = \frac{1}{m} T_i. \quad (7)$$

The normalized supporting element $l_i = F_i$ can be written in terms of h as:

$$l_i = \frac{h_{i0}}{F^{m-1}}. \quad (8)$$

Remark 1 In [14], [3], one uses the homogenized version

$$a_{ij} = \frac{1}{F^{m-2}} h_{ij}. \quad (9)$$

The link between h_{ij} and the usual Finsler metric g_{ij} is given by:

$$h_{ij} = F^{m-2} \left(\frac{1}{m-1} g_{ij} + \frac{m-2}{m-1} l_i l_j \right). \quad (10)$$

Taking into account that g_{ij} is nondegenerate on $TM \setminus \{0\}$, it follows that h_{ij} itself is nondegenerate on $TM \setminus \{0\}$; its inverse matrix is:

$$h^{ij} = \frac{1}{F^{m-2}} \{ (m-1) g^{ij} - (m-2) l^i l^j \}.$$

In other words, h is a Lagrange metric, [11], on M .

Moreover, if g is positive-definite, then so is the polynomial metric h .

The proof of both nondegeneracy and positive-definiteness, as well as the computation of the inverse matrix h^{ij} rely on 10 and the following lemma ([5]):

Lemma 2 Consider the following family of $(0,2)$ -tensor fields:

$$\Theta_{ij} = \lambda g_{ij} + \mu l_i l_j, \quad \lambda, \mu \in \mathcal{F}(M).$$

Denote by g^{ij} the dual of g_{ij} . Then

- a) Θ_{ij} is non-degenerate for $\lambda(\lambda + \mu) \neq 0$ on TM ;
- b) The dual of Θ_{ij} is

$$\Theta^{ij} = \frac{1}{\lambda} g^{ij} + \frac{-\mu}{\lambda(\lambda + \mu) F^2} y^i y^j;$$

- c) the determinant of Θ is $\det \Theta = \lambda^{n-1}(\lambda + \mu) \det g$.

By expressing the Euler-Lagrange equations attached to F in terms of h_{ij} , we get

Proposition 3 The equations of unit-speed geodesics $t \mapsto (x^i(t))$ of an m -th root metric space can be expressed in terms of the polynomial metric h_{ij} as

$$\frac{d^2 x^i}{dt^2} + \gamma^i_{j_1 \dots j_m} y^{j_1} \dots y^{j_m} = 0, \quad (11)$$

where $y^i = \frac{dx^i}{dt}$ and the generalized Christoffel symbols $\gamma^i_{j_1 \dots j_m}$ are

$$\gamma^i_{j_1 \dots j_m} = \frac{h^{ip}}{m(m-1)} \left(\sum_{(j_1, \dots, j_m)} \frac{\partial a_{pj_2 \dots j_m}}{\partial x^{j_1}} - \frac{\partial a_{j_1 \dots j_m}}{\partial x^p} \right), \quad (12)$$

with $\sum_{(j_1, \dots, j_m)}$ denoting cyclic sum w.r.t. the involved indices.

Obviously, $\gamma^i_{j_1 \dots j_m}$ are totally symmetric w.r.t. the lower indices. Moreover,

$$\gamma_{pj_1 \dots j_m} = h_{ip} \gamma^i_{j_1 \dots j_m}$$

depend only on x .

We notice that

$$\gamma_{p00 \dots 0} = \gamma_{pj_1 \dots j_m} y^{j_1} \dots y^{j_m}.$$

are polynomial functions (homogeneous of degree m) in y^i .

According to [11], the canonical spray coefficients of (M, F) are

$$2G^i = \gamma^i_{00 \dots 0}. \quad (13)$$

It is also useful to express the canonical (Kern) nonlinear connection and the Berwald connection of the given Finsler space in terms of h_{ij} . Namely, the coefficients of the nonlinear connection are

$$N^i_j = G^i_{\cdot j} = \frac{1}{2} \frac{\partial}{\partial y^j} (h^{is} \gamma_{s \ 00 \dots 0}) = \frac{1}{2} (h^{is}_{\cdot j} \gamma_{s \ 0 \dots 0} + m h^{is}_{\cdot j} \gamma_{s \ j 0 \dots 0}). \quad (14)$$

In the following, we shall always mean by N the Kern connection (14).

Further, the Berwald connection coefficients $G^i_{jk} = G^i_{\cdot jk}$, [1], [6], are:

$$G^i_{jk} = \frac{1}{2} (h^{is}_{\cdot jk} \gamma_{s \ 0 \dots 0} + m h^{is}_{\cdot j} \gamma_{s \ k 0 \dots 0} + m h^{is}_{\cdot k} \gamma_{s \ j 0 \dots 0} + m(m-1) h^{is}_{\cdot jk} \gamma_{s \ jk 0 \dots 0}). \quad (15)$$

By subsequent derivation, we infer that its hv -curvature $G^i_{jkl} := G^i_{jk \cdot l}$ is given by

$$G^i_{jkl} = \frac{1}{2} \{ h^{is}_{\cdot jkl} \gamma_{s \ 0 \dots 0} + m \sum_{(j,k,l)} (h^{is}_{\cdot jk} \gamma_{s \ l 0 \dots 0}) + m(m-1) \sum_{(j,k,l)} (h^{is}_{\cdot j} \gamma_{s \ kl 0 \dots 0}) + m(m-1)(m-2) h^{is}_{\cdot jk} \gamma_{s \ jkl 0 \dots 0} \}. \quad (16)$$

Notes: 1) The equations of geodesics (11) can also be expressed as

$$\frac{d^2 x^i}{dt^2} + G^i = 0 \Leftrightarrow \frac{d^2 x^i}{dt^2} + N^i_j y^j = 0 \Leftrightarrow \frac{d^2 x^i}{dt^2} + L^i_{jk} y^j y^k = 0.$$

2) The Kern nonlinear connection and the Berwald connection $B\Gamma(N)$ depend only on the function F , *not* on the choice of the metric tensor $(h_{ij}, a_{ij}$ or $g_{ij})$. We cannot say the same of the canonical metrical connection $CT(N)$.

By *canonical metrical connection* $CT(N)$ attached to a generalized Lagrange metric α_{ij} , [11], we mean the normal connection (L^i_{jk}, C^i_{jk}) given by

$$\begin{aligned} L^i_{jk} &= \frac{1}{2} \alpha^{ih} (\alpha_{hj;k} + \alpha_{hk;j} - \alpha_{jk;h}) \\ C^i_{jk} &= \frac{1}{2} \alpha^{ih} (\alpha_{hj \cdot k} + \alpha_{hk \cdot j} - \alpha_{jk \cdot h}). \end{aligned}$$

In particular, if N is the Kern nonlinear connection and α_{ij} is a Finsler metric, then $(N^i_j, L^i_{jk}, C^i_{jk})$ is the Cartan connection.

3) For both $B\Gamma(N)$ and $CT(N)$ there holds, [11]:

$$y^i|_k = 0,$$

where $|$ denotes the horizontal covariant derivative.

3 m-th root metric spaces of Landsberg and Berwald type

A Finsler space (M, F) is called a *Landsberg space* if the Berwald connection $B\Gamma(N)$ is h -metrical. Equivalently, [1], (M, F) is a Landsberg space iff the horizontal coefficients of $B\Gamma(N)$ and $CT(N)$ coincide:

$$G^i_{jk} = L^i_{jk}, \quad \forall i, j, k = 1, \dots, n.$$

(M, F) is called a *Berwald space* if the coefficients G^i_{jk} are functions of position x alone. Equivalently: $G^i_{jkl} = 0$.

There holds the inclusion, [1]:

$$\text{Berwald spaces} \subset \text{Landsberg spaces}.$$

In [14], there is proven that

1. The horizontal coefficients of the canonical metrical connection ${}^aCT(N)$ attached to the homogenized metric a_{ij} (9) coincide with those of the Cartan connection (attached to the usual Finsler metric) ${}^*CT(N)$ of (M, F) ;

2. an m -th root metric space (M, F) is a Berwald space (resp. Landsberg space) if and only if, w.r.t. ${}^aCT(N)$, we have $a_{ijk|h} = 0$ (resp. $a_{ijk|0} = 0$, where

$$a_{ijk} = \frac{a_{ijk00\dots 0}}{F^{m-3}}.$$

Now, by using:

$$\begin{aligned} h_{ij} &= F^{m-2}a_{ij}, \quad F_{,l} = 0, \\ h_{ij\cdot k} &= (m-2)F^{m-3}a_{ijk}, \end{aligned}$$

we can immediately see that there holds

Proposition 4 *The horizontal coefficients L^i_{jk} of the canonical metrical connection ${}^hCT(N)$ attached to the polynomial metric h coincide with those of the Cartan connection ${}^*CT(N)$ of (M, F) .*

Corollary 5 *An m -th root metric space (M, F) is a Berwald space (resp. Landsberg space) if and only if, w.r.t. the canonical metrical connection ${}^hCT(N)$, we have $h_{ij\cdot k|l} = 0$ (resp. $h_{ij\cdot k|0} = 0$).*

For Landsberg spaces, we can obtain easier conditions if we use the Berwald connection instead of the canonical metrical one.

Namely, by using (10), (8) and the equalities $y^i_{|k} = 0$, we get:

Theorem 6 *An m -th root metric space (M, F) is of Landsberg type if and only if, with respect to the Berwald connection $B\Gamma(N)$, we have:*

$$h_{ij|k} = 0, \quad i, j, k = 1, \dots, n. \quad (17)$$

4 Finsler metrics in projective relation

Definition. ([1]) Let $F^n = (M^n, F(x, y))$ and $\bar{F}^n = (M^n, \bar{F}(x, y))$ be two Finsler spaces on the same underlying manifold M^n . If any geodesic of F^n coincides with a geodesic of \bar{F}^n as a set of points and vice versa, then the change $F \rightarrow \bar{F}$ is called *projective* and F^n is said to be *projective to \bar{F}^n* .

In the following, we shall expose several classical results referring to projective changes:

Theorem 7 (Knebelmann), [1], [2]: A Finsler space F^n is projective to another Finsler space \bar{F}^n if and only if there exists a 1-homogeneous scalar field $p(x, y)$ obeying

$$\bar{G}^i = G^i + p(x, y)y^i \quad (18)$$

A very useful characterization of projectivity is also given by Ràpcsák's theorem :

Theorem 8 (Ràpcsák), [1]: A Finsler space $F^n = (M^n, F(x, y))$ is projective to $\bar{F}^n = (M^n, \bar{F}(x, y))$ if and only if \bar{F} satisfies one of the following three equations:

1. $\bar{F}_{;j} - y^r \bar{F}_{;r,j} = 0$;
2. $\bar{l}_{j;i} - \bar{l}_{i;j} = 0$;
3. $\bar{F}_{;i,j} - \bar{F}_{;j,i} = 0$, where $;$ means the covariant derivative in F^n .

Definition 9 A Finsler space F^n is said to be projectively flat if its geodesics are straight lines.

There holds

Theorem 10 [1], [2]: A Finsler space is projectively flat iff it is projective to a locally Minkovski space.

For Berwald-type projectively flat spaces, we have the following characterization:

Theorem 11 ,[2], A Finsler space F^n (of dimension n) is a projectively flat Berwald space if and only if it belongs to one of the following classes:

1. $n \geq 3$:
 - a) locally Minkovski spaces;
 - b) Riemannian spaces of constant curvature;
2. $n = 2$:
 - a) locally Minkovski spaces;
 - b) Riemannian spaces of constant curvature;
 - c) spaces F^2 with $F = \frac{\beta^2}{\gamma}$, where β and γ are 1-forms.

Obviously, in the case of m-th root metric spaces, we cannot have the situation 2c), which implies that the only m-th root metric spaces which are both Berwald and projectively flat are either locally Minkovski, either Riemannian ($m = 2$) of constant curvature.

Another important category of Finsler spaces are *Douglas spaces*.

Definition 12 A Finsler space is said to be of Douglas type, or a Douglas space, if

$$D^{ij} = G^i y^j - G^j y^i \quad (19)$$

are homogeneous polynomials in y^i of degree three.

A characterization of Douglas spaces is given by

Theorem 13 [2]: A Finsler space is of Douglas type if and only if the Douglas projective tensor

$$D^i_{jkl} = G^i_{jkl} - G_{jk \cdot l} y^i / (n+1) - (G_{jk} \delta_l^i + G_{lj} \delta_k^i + G_{kl} \delta_j^i) / (n+1), \quad (20)$$

(where $G_{jk} = G^i_{jki}$) vanishes identically.

The Douglas tensor D^i_{jkl} is invariant under projective transformations.

Both locally Minkovski and Berwald (including Riemannian) spaces are Douglas spaces.

Here are several results related to Douglas spaces, [2]:

Theorem 14 Any Finsler space projective to a Douglas space is itself a Douglas space.

Theorem 15 F^n is a Berwald space iff it is a Landsberg space and a Douglas space.

Finally, the most important to us:

Theorem 16 For positive-definite Finsler manifolds, there holds the inclusion: **projective to Riemann = projective to Berwald \subset Douglas.**

The equality in the theorem above is due to Szabó's theorem of metrizability of (positive definite) Berwald manifolds, [6], [17], [2].

5 Projectively flat m-th root metric spaces

In order to study projective flatness for m-th root metric spaces, we use the condition (1) in Ràpcsák's theorem:

$$\bar{F}_{;j} - y^r \bar{F}_{;r \cdot j} = 0.$$

Let, for the moment, \bar{F} denote an m-th root metric, and F , a locally Minkovski one.

Since (M^n, F) is locally Minkovski, in a certain coordinate system we have $N^i_j = 0$, this is, $\bar{F}_{;i} = \bar{F}_{;i}$ and the mentioned condition can be written by using only usual partial derivatives:

$$\bar{F}_{;j} - y^r \bar{F}_{;r \cdot j} = 0. \quad (21)$$

In terms of $T = \bar{F}^m$, we obtain that the last condition is equivalent to

$$T(T_{,j} - y^r T_{,r,j}) = y^r (\frac{1}{m} - 1) T_j T_{,r}. \quad (22)$$

But

$$T_{,j} - y^r T_{,r,j} = -m(m-1)\gamma_{j0\dots 0}, T_j = mh_{j0}, T = h_{00}; \quad (23)$$

consequently, we have:

Theorem 17 *An m -th root metric space is projectively flat if and only if*

$$mh_{00}\gamma_{j0\dots 0} = h_{j0}y^r h_{00,r}.$$

Both terms in the above relation are polynomials in y^i .

By replacing the terms in the above equality with their expressions in $a_{i_1\dots i_m}$ and by identifying the corresponding coefficients, it follows

Theorem 18 *A m -th root metric space is projectively flat if and only if*

$$m \sum a_{ri_2\dots i_m} \gamma_{j \ j_1\dots j_m} = \sum a_{ji_2\dots i_m} a_{j_1\dots j_m, r}, \quad (24)$$

$\forall j, r, i_1, \dots, i_m, j_1, \dots, j_m = \overline{1, r}$ and the symbol \sum means symmetrization with respect to $i_1, \dots, i_m, j_1, \dots, j_m$.

6 m-th root metric spaces projectively related to Riemannian spaces

Throughout this section, we shall need to assume that F is properly Finslerian, i.e., (g_{ij}) in (5) is *positive definite*. As a remark, the results above hold true also for the case of (nondegenerate) g_{ij} of arbitrary signature.

As said above, by Szabó's theorem, a (positive-definite) Finsler space is projective to a Riemannian space if and only if it is projective to a Berwald one.

A necessary condition for F^n to be Berwald-projective is that F^n should be of Douglas type. Consequently, a necessary condition for a m -th root metric space to be Berwald-projective is that its Douglas tensor should vanish:

$$D^i_{jkl} = 0.$$

A condition which is also sufficient is obtained by starting with condition 1 in Rapcsák's theorem.

Let $\Gamma^i_{jk}(x)$ be the Berwald connection coefficients of some Berwald space (M, \tilde{F}) , $N^i_j = \Gamma^i_{j0}$ the corresponding nonlinear connection and let " ; " denote the associated covariant derivative.

In terms of $T = F^n$, the cited condition writes:

$$mT(T_{;i} - T_{;r,i}y^r) = (1-m)T_{;r}T_iy^r. \quad (25)$$

(25) is an equality of homogeneous polynomials of degree $2m$ in y^i .

By a direct computation of the involved derivatives, it follows that (25) is equivalent to

$$mT(T_{,i} - T_{,r}y^r) + (m-1)T_iT_{,r}y^r = -\Gamma_{00}^i\{(1-m)T_iT_j + mTT_{ij}\}. \quad (26)$$

Unfortunately, the matrix with the entries

$$A_{ij} = (1-m)T_iT_j + mTT_{ij}$$

is degenerate, so we cannot use its inverse in order to separate variables in the right-hand side. Though, taking into account that in both sides we have polynomials, we can simply identify their coefficients. What we obtain is a linear system in Γ_{jk}^i . Namely, we have

Theorem 19 *The m -th root metric space F^n is Riemann-projective iff there exist the functions $\Gamma_{jk}^i(x)$ which obey (26) and, w.r.t. coordinate changes on M , they obey the rules of transformation of the coefficients of a linear connection.*

By replacing the derivatives of T in (26) and identifying the coefficients, it follows:

Theorem 20 *The m -th root metric space F^n is Riemann-projective iff there exist the functions Γ_{jk}^i which obey*

$$\begin{aligned} & a_{j_1 \dots j_m} \gamma_{i \ i_1 \dots i_m} - \frac{1}{m} a_{ij_2 \dots j_m} \cdot a_{i_1 \dots i_m}, j_1 = \\ & = \Gamma_{i_1 j_1}^j \{-a_{ii_2 \dots i_m} a_{jj_2 \dots j_m} + a_{ij i_3 \dots i_m} a_{i_1 i_2 j_3 \dots j_m}\}. \end{aligned} \quad (27)$$

and, w.r.t. coordinate changes on M , they obey the rules of transformation of the coefficients of a linear connection.

7 The transformation $\bar{T} = \alpha(x, y)T$

This type of transformation allows to reduce the order of the metric, in the following sense: if \bar{T} defines an m -th root metric as in (4), then, we are trying to find some α such that (M^n, \bar{T}) be projective to a m -th root metric space whose metric has smaller root order (M^n, T) (in particular, to a space of order two, this is, to a Riemann space, where $T = a_{ij}(x)y^i y^j$).

Let T and \bar{T} yield two m -th root metrics of order m' and m respectively, related by

$$\bar{T} = \alpha(x, y)T.$$

Suppose $m' < m$. Then, α is homogeneous of order $p = m - m' > 0$ w.r.t. y . Let again N_j^i denote the coefficients of the Cartan connection attached to $F = T^{1/m'}$.

By means of (25):

$$\bar{T}_{;i} - \bar{T}_{;r \cdot i} y^r = \frac{1-m}{m\bar{T}} \bar{T}_{;r} \bar{T}_i y^r. \quad (28)$$

and of the equality $T_{;i} = 0$, we obtain that T and \bar{T} are projectively related iff

$$\alpha_{;i} - \alpha_{;r \cdot i} y^r = \frac{1}{m} \frac{T_i}{T} \alpha_{;0} + \frac{1-m}{m} \frac{\alpha_i}{\alpha} \alpha_{;0}. \quad (29)$$

Remark: If $\alpha = \alpha(x)$ (conformal transformations), (29) is written as:

$$\alpha_{;i} = \frac{1}{m} \frac{T_i}{T} \alpha_{,r} y^r,$$

or

$$\alpha_{,r} (\delta_i^r - \frac{1}{m} \frac{T_i}{T} y^r) = 0.$$

We now return to $\alpha = \alpha(x, y)$. Let us notice that any α with vanishing covariant derivative $\alpha_{;i} = 0$ obeys (29). As a consequence, we get

Proposition 21 *Let $\gamma = (\gamma_{ij}(x))$ be a Riemannian metric on M and $\alpha = \alpha(x, y)$ a polynomial function in y (of degree $m > 0$) with the property*

$$\alpha_{;i} = 0, \quad (30)$$

where the covariant derivative is taken w.r.t. the canonical linear connection (14) arising from γ . Then, the m -th root metric space (M, F) with

$$F^{m+2} = \bar{T} = \alpha\gamma \quad (31)$$

is projective to the Riemannian space (M, γ) .

In particular, above we consider a projectively flat Riemannian metric γ , we get

Proposition 22 *If γ in (31) is a projectively flat Riemannian metric, then \bar{T} in (31), (30) is a projectively flat m -th root metric.*

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